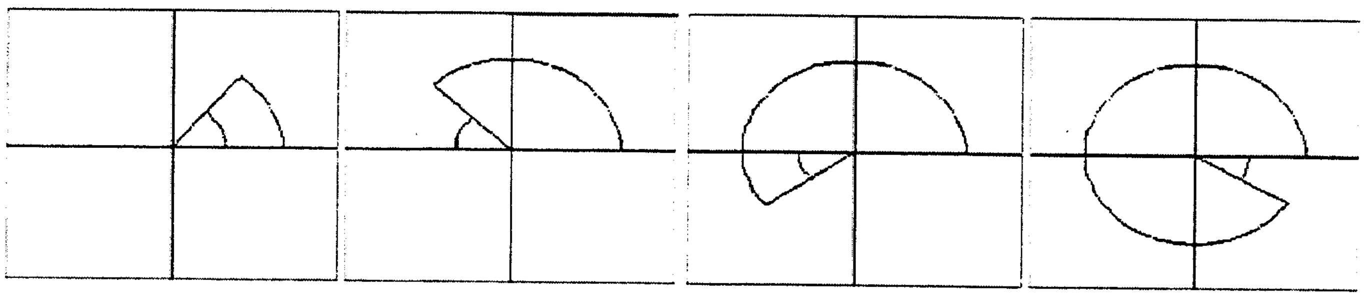
Trigonometry: Reference Angles

The repetitions and patterns of the trigonometric functions make many calculations easier. Instead of memorizing many angles and the values of the trigonometric functions for each of the angles, we can use the idea of reference angles to cut down on the number of angles we need to look at. Reference angles are defined as the *acute* (less than 90°) angle that a given angle makes with the x-axis (positive or negative). **As these graphs show, the reference angle in the first quadrant is itself; in the second quadrant, is what is missing from one-half revolution; in the third quadrant, it is what is extra from one-half revolution; and in the fourth quadrant, it is what is missing from one whole revolution.**



In each of these four graphs, the outer arc is the original angle, and the inner arc is the reference angle. Where does this get us? The major value of reference angles comes from the fact that for any trigonometric function, we can exchange an angle and its reference angle, and the answer is exactly the same, as long as we give it the correct sign. How do we know which sign to give it? That comes from knowing where the functions are positive and where they are negative. This can be remembered easily by using the mnemonic "**All Students Take Calculus**" —starting in the first quadrant, go counter-clockwise, assigning each letter to each quadrant. You can also use the word CAST. Using CAST, start in the *fourth* quadrant. Either way, we have:

S	**A**
T	**C**

A stands for "all", which means that all the functions are positive in the first quadrant. S stands for "sine", which means that only sine is positive in the second quadrant. T is for "tangent", and C is for "cosine".

Numerically, the way to find a reference angle for some angle, θ, is as follows: in quadrant I, it is the same as θ. In quadrant II, it is equal to $180 - \theta$. In quadrant III, it is $\theta - 180$. Finally, in quadrant IV, it is $360 - \theta$.

So, as an example, let's look at finding tangent 135°. This is not a memorized value, so let's try using reference angles. 135° is in the second quadrant, so our reference angle is 180°-135°, or 45°. We know that the tangent of 45° is 1, but is it positive or negative? This is where we use the CAST rule. In quadrant two, only sine is positive, so tangent is negative. Therefore, tan (135°) = -1.

Reference angles also come into play when we are evaluating inverse trigonometric functions. Your calculator may give you an answer to these, but it is not always the correct answer. This is because for a function to have an inverse, it must be one-to-one, but some applications of trigonometry are not limited in this way. So, you may be given the following: $\sin(x) = -\sqrt{2}/2$ and asked to find x, where x is in the third quadrant. Your calculator will tell you that $\sin^{-1}(-\sqrt{2}/2) = -45°$, but this is not in the third quadrant. Because the calculator gave us -45°, we know that the reference angle is 45°, so we need to find what angle in the third quadrant has a reference angle of 45°. Using the rules of reference angles, we see that it is 225°. If we check on the calculator, $\sin(225°)$ is in fact $-\sqrt{2}/2$.