Binomial Probability

If the distribution of a random variable, x, fulfills the following requirements, then it is referred to as a "binomial distribution." We will learn how to compute binomial probabilities and characteristics of the binomial distribution.

There are four requirements for a binomial experiment:

1. Each trial must have exactly TWO categories for outcomes (success and failure).
2. Experiment must have a fixed number of trials. (n)
3. Each trial must be independent of the others.
4. The probabilities for each trial must remain constant.

p = probability of a success
q = probability of a failure
Success and failure are complementary events so p + q = 1.

The binomial formula states...

$$P(x) = {}_nC_x \cdot p^{x} \cdot q^{n-x}$$

n is the number of trials (must be fixed).
x is the number of successes out of n trials (remember that x may be any whole number between 0 and n, inclusive: x = 0, 1, 2 ... n).
p is the probability of success on any given trial.
q is the probability of failure on any given trial (q = 1 – p).
P(x) is the probability of getting exactly x successes out of n trials.

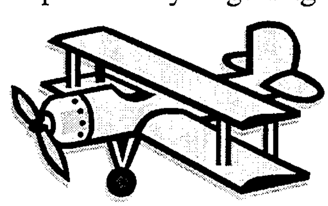
${}_nC_x$ should be done on your calculator if possible. If not . . . ${}_nC_x = \dfrac{n!}{(n-x)!\,x!}$

Let's try an example.

A husband and wife want to have 6 kids. Girls and boys are equally likely. What is the probability of the couple having more than 4 girls?

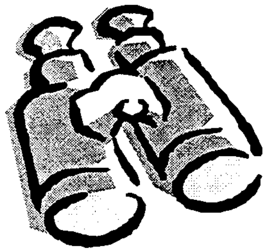
First identify if we have a binomial. Check to see if the experiment satisfies the four requirements listed above.

Now identify the given information that we have. How many times will the experiment be repeated? This is n. What is the probability of a "success" on a given trial? This is p. If we know p, we can find q. Remember, success and failure are complements, so q = 1 – p. In this problem, n = 6, since that is the number of trials. A "success" will be the couple having a girl, since the question is asking about getting a certain number of girls. This means that p = P(girl on any given trial) = 0.5, since girls and boys are equally likely (which also implies that q = P(boy on any given trial) = 0.5). We are looking for more than 4 girls. "More than 4" does not include the 4. This means that the number of successes, x, must be 5 and 6. We will have to find the probability of getting exactly 5 girls and add that to the probability of getting exactly 6 girls.



P(more than 4 girls) = P (x > 4) = P(exactly 5 girls) + P(exactly 6 girls)

$= P(x=5) + P(x=6)$



$= {}_6C_5\,(0.5)^5(0.5)^1 + {}_6C_6\,(0.5)^6(0.5)^0$

$= (6)(0.03125)(0.5) + (1)(0.015625)(1)$

$= 0.09375 + 0.015625$

$= 0.109375$

$\approx$ (0.109)

Substituting the values we know into the binomial formula.
$n = 6$
$p = 0.5$
$q = 0.5$
$x = 5, 6$

Let's say the same couple uses some fertility drug to increase their chances of having a girl from 0.5 to 0.58. Instead of 6 kids, they decide to have 4, and they want to know the probability of having at least one girl.

"At least one" includes the outcome of one girl, two girls, three girls and four girls, so we could find the probability here by adding the probabilities for each of these outcomes.

$n = 4$
$p = 0.58$
$q = 1 - 0.58 = 0.42$

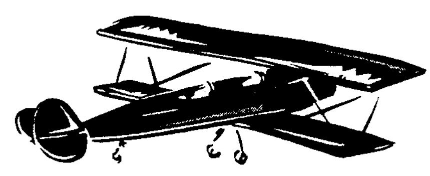
P(at least 1 girl) = $P(x \geq 1) = [P(x=1) + P(x=2) + P(x=3) + P(x=4)]$

$= {}_4C_1(0.58)^1(0.42)^3 + {}_4C_2(0.58)^2(0.42)^2 + {}_4C_3(0.58)^3(0.42)^1 + {}_4C_4(0.58)^4(0.42)^0$

$\approx 0.17188 + 0.35605 + 0.32779 + 0.11316$

$\approx$ (0.969)

This is a bit tedious. There is another way to do this problem. We could look for the complement of "at least one girl." The rule of complements states that the probability of event A plus the probability of the complement of A must equal one. The complement of A consists of all events that are not A and is often denoted with $\overline{A}$.



Rule of complements

$P(A) + P(\overline{A}) = 1$ or $P(A) = 1 - P(\overline{A})$ or $P(\overline{A}) = 1 - P(A)$

So, what is the complement of "at least one girl?" It must be "NOT at least one girl", which would imply "all non-girls" or better said, "all boys." Applying the rule of complements to this problem, we get the following.

P(at least one girl) = 1 – P(all boys)

$P(x \geq 1) = 1 - P(x < 1) = 1 - P(x = 0)$

$= 1 - {}_4C_0(0.58)^0(0.42)^4$

$= 1 - 0.03111696$

$\approx$ (0.969)

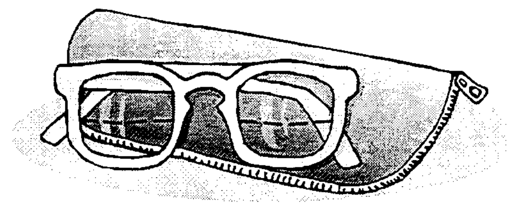
This gives us the correct answer and we only had to use the binomial formula once. This same technique can be applied to any probability problem, but it is especially useful when asked about "at least one." The complement of "at least one" is always easy to find. The complement of "at least one success" is always "all non-successes" or "all failures."

Let's try another example.

A quiz for a Psychology class consists of 5 questions. The questions are multiple-choice with 4 possible answers each. A student randomly guesses on all 5 questions.

A.) What is the probability of that student getting a passing grade (60% correct is passing)?
B.) What is the probability of that student getting a B or better (a low B is 80%)?
C.) What is the probability of that student failing the quiz?

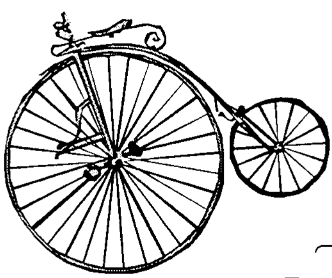
Since this problem satisfies the four requirements for a binomial experiment, we can use the binomial formula. Let's identify some of the information we need. There are five trials (five questions) so n = 5. Since we are concerned with the number of problems the student gets correct, a success will be a correct guess. There are four possible answers for each test question and only one of these is the correct answer, so the probability of the student guessing correctly on any given trial is ¼ or 0.25 (p = 0.25). This means that the probability of an incorrect guess (a failure) is 1 – 0.25 = 0.75 (q = 0.75). The only other piece of information we need is the number of successes we are looking for. These values for "x" are different for the different parts of this question.



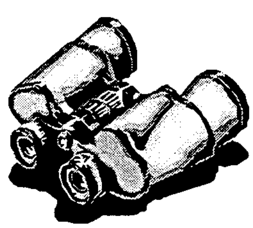
$n = 5$
$p = 0.25$
$q = 0.75$

Since there are five questions, a passing grade would mean at least 60% of those five questions were correct. This means that a score of 3 or better would be passing. So, to find this probability we will add up the probabilities for 3 correct, 4 correct and 5 correct.

A.) Probability of passing = P(getting 60% or better)
= P(x ≥ "60% of 5")
= P(x ≥ 0.6*5)
= P(x ≥ 3)
= P(x = 3) + P(x = 4) + P(x = 5)



$$= {}_5C_3\,(0.25)^3\,(0.75)^2 + {}_5C_4\,(0.25)^4\,(0.75)^1 + {}_5C_5\,(0.25)^5\,(0.75)^0$$
$$\approx 0.087891 + 0.014648 + 0.00097656$$
$$\approx 0.104$$



A note about rounding: The generally accepted rule for rounding probabilities is to take three significant figures, but be careful about rounding intermediate values (values that you arrive at before your final answer) as this could introduce error into your answer. Either avoid rounding intermediate values all together or take four or five significant figures for these intermediate values. Then round the final answer to three significant figures.

B.) Probability of getting a B or better = P(80% or better) = P(x ≥ 0.8*5) = P(x ≥ 4) = P(x = 4) + P(x = 5)
= 0.014648 + 0.00097656 = 0.0156

Here, we used the values that we determined in part A.

C.) Probability of failing = Probability of not passing = P(not getting 60% or better)

$= 1 - P(\text{passing})$

$= 1 - P(x \geq 3)$

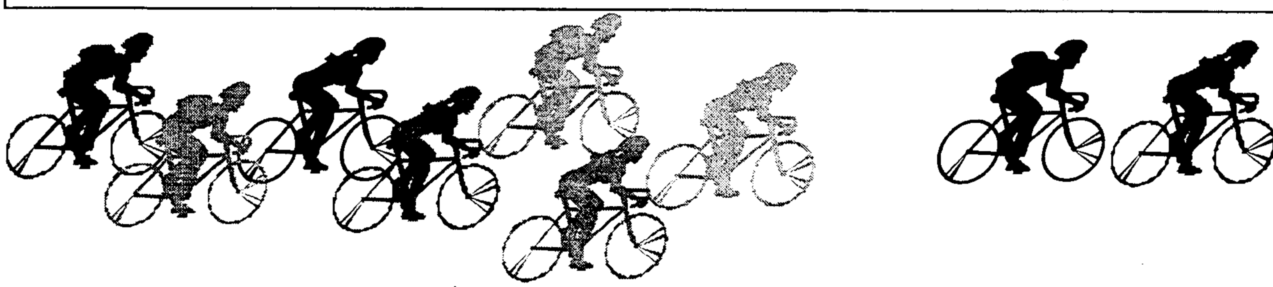
$= 1 - [0.087891 + 0.014648 + 0.00097656]$

$= 1 - 0.10352$

$= \boxed{0.896}$

Since "passing and "failing" are complementary events, we can use the rule of complements to find the probability of failing. P(failing) = 1 – P(passing) = 1 – P(at least 60%) = 1 – P(x ≥ 3). The values we use for P(3), P(4), and P(5) are taken from part A.

This is a pretty high likelihood, which means that if the student randomly guesses on this quiz, (s)he will probably receive a failing grade. Of course, if the student studies and his/her answers are not random guesses, (or even if they can narrow the choices down to two instead of four), the probability of failing will decrease and the student will have a better chance of passing the quiz.



Just in case you need them, the equations to find mean, standard deviation and variance for a binomial are quite simple:

mean = $\mu = np$ standard deviation = $\sigma = \sqrt{npq}$ variance = $\sigma^2 = npq$

Let's try an example that employs these equations.

In the game of craps (involving rolling a pair of dice), there is a 1/6 chance of rolling a seven. (You may verify this by using the rules for basic probabilities – see the resource sheet on "Probability.")

A.) The dice are rolled 35 times; find the mean and the standard deviation for the number of sevens rolled.
B.) Would it be unusual for me to get 12 sevens out of my 35 rolls?

A.) $\mu = np = (35)\left(\frac{1}{6}\right) \approx \boxed{5.8333}$ $\sigma = \sqrt{npq} = \sqrt{(35)\left(\frac{1}{6}\right)\left(\frac{5}{6}\right)} \approx \boxed{2.2048}$

B.) A score is unusual if it is more than 2 standard deviations away from the mean. Find $\mu - 2\sigma$ and $\mu + 2\sigma$. If 12 is within these two values, it is considered "usual," otherwise, it is "unusual."

$\mu - 2\sigma = 5.8333 - 2(2.2048) = 1.4237$
$\mu + 2\sigma = 5.8333 + 2(2.2048) = 10.2429$

Since 12 is not within 2 standard deviations of the mean (12 is not in between 1.4237 and 10.2429) we conclude it to be <u>unusual</u> to get 12 sevens out of 35 rolls.