Binomial Theorem

If you had to expand $(x-3y)^2$, then simple FOIL. But something like $(2x-4)^{12}$, would take a very long time to expand. There are two methods that may be used to accomplish this task. One of them is called Pascal's triangle but is only used if the exponent is less than 9. The other method is considered to be the most efficient. The Binomial Theorem which looks like this:

- $(a+b)^n = \binom{n}{0}a^n b^0 + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \binom{n}{3}a^{n-3}b^3 + \ldots + \binom{n}{n-1}ab^{n-1} + \binom{n}{n}a^0 b^n$

This formula works for any binomial $(a+b)$ and any natural number *n* = 1,2,3, ...

The definition of the lead term (called *n* choose *k)* is: $\binom{n}{k} = \frac{n!}{(n-k)!k!}$ where the factorial n! is an operation that multiplies all natural numbers n down to 1. For example $5! = 5\cdot 4\cdot 3\cdot 2\cdot 1 = 120$. Also, by definition 0! = 1.

Example: $\binom{7}{3} = \frac{7!}{(7-3)!3!} = \frac{7!}{4!3!} = \frac{7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1}{(4\cdot 3\cdot 2\cdot 1)(3\cdot 2\cdot 1)} \Leftarrow$ Reduce first $\Rightarrow \frac{7\cdot 5}{1} = 35$

Note: This can also be found by your calculator by keying in: 7 $\boxed{{}_nC_r}$ 3 = 35

If you look closely at the theorem, you'll notice that the exponent on *a* decreases with each term and the exponent on *b* increases. So the first term will have a^n then count down until you reach a^0 (**remember** $a^0 = 1$) The reverse is true for the *b* term.

- Example: expand $(2x-3y)^5$ by following the pattern rather than using the formula.

First note that ***a = 2x*** and ***b = –3y*** (include the negative) and that the exponent ***n = 5***

Start with the first term: $\binom{5}{0}(2x)^5(-3y)^0 + \ldots$

- $\binom{5}{0}$ The top will be *n* = 5 and the bottom will start at zero and count up to 5 for each term.
- $(2x)^5$ The *a* term will start with the power of 5 and countdown to zero for each term.
- $(-3y)^0$ The *b* term will start with the power of 0 and count up to 5 for each term.

$$\binom{5}{0}(2x)^5(-3y)^0 + \binom{5}{1}(2x)^4(-3y)^1 + \binom{5}{2}(2x)^3(-3y)^2 + \binom{5}{3}(2x)^2(-3y)^3 + \binom{5}{4}(2x)^1(-3y)^4 + \binom{5}{5}(2x)^0(-3y)^5$$

Simplify. Remember anything to the zero power is one. Use the $\boxed{{}_nC_r}$ on your calculator to evaluate all the $\binom{n}{k}$ terms.

- In order, they evaluate to: 1, 5, 10, 10, 5, 1 (which is the 5^{th} line of Pascal's triangle)
- When evaluating these terms notice that the last half of the list is a repeat of the first half.

Evaluate each term first. Do you remember order of operations? Parenthesis – Exponents – Etc.

Also, a negative number raised to an odd power results in a negative number $(-3y)^3 = -27y^3$

$$1(32x^5)(1) + 5(16x^4)(-3y) + 10(8x^3)(9y^2) + 10(4x^2)(-27y^3) + 5(2x)(81y^4) + 1(1)(-243y^5)$$

Now multiply each term together.

$$32x^5 - 240x^4y + 720x^3y^2 - 1080x^2y^3 + 810xy^4 - 243y^5$$

One thing to notice is that the number of terms is one more than the exponent. See here that we ended up with 5 + 1 = 6 terms. Also, notice that ***b = –3y*** is negative and the signs in the polynomial alternate from plus to minus with each term.

Single term of an expansion

The Binomial theorem can be used to find a single term of an expansion. For instance, suppose you have $(2x+y)^{12}$. The question may only ask to find the 5^{th} term of the polynomial. In that case we just want to use the formula below.

- $\binom{n}{k-1}a^{n-(k-1)}b^{k-1}$ Where k equals the term number.

So, like before we find ***a = 2x***, ***b = y*** , ***n = 12*** and ***k = 5*** (for the fifth term)

- $\binom{12}{5-1}(2x)^{12-(5-1)}(y)^{5-1}$ $\Leftarrow$ enter values into the formula
- $\binom{12}{4}(2x)^8(y)^4$ $\Leftarrow$ Simplify each power first
- $495(2)^8x^8y^4$ $\Leftarrow$ evaluate the lead term, $12\ \boxed{{}_nC_r}\ 4$
- $126720x^8y^4$ $\Leftarrow$ Multiply the numbers. Done