Sequences and Series

Patterns in numbers turn up everywhere – not just in math, but also in nature. Two types of patterns that are of interest are arithmetic and geometric sequences and series. A sequence is a list of numbers, such as 1, 2, 3, 4. They may stop at some point or continue on forever. A series is like a sequence, except we add the numbers, instead of just listing them. All the formulas and rules for finding terms of sequences are the same as for series, so if we find something that is true for one, we know it is also true for the other.

Arithmetic Sequences

An arithmetic sequence is one in which there is a common difference between one term and the next. An example would be 3, 6, 9, 12… Each term is 3 more than the term before it. Another example would be 4, 7, 10, 13… Again, each term is 3 more than the one before it, but we started at a different number. These are both arithmetic sequences, because there is a common difference. We label the terms in the series or sequence with subscripts, so that the first term is a_1, the second is a_2, etc. Thus, the $n^{\underline{th}}$ term is a_n. For sequences, the sum of the first n terms is denoted by S_n. So S_{21} would be the sum of the first 21 terms. If we want to know a particular term in the sequence, there are two things we have to know: the first term, which we call a_1, and the common difference, d. The formula for finding the $n^{\underline{th}}$ term is given by:

$$a_n = a_1 + (n-1)*d$$

If we want to know the sum of the first n of these terms, we can use the formula

$$S_n = \frac{n}{2}(a_1 + a_n)$$

Geometric Sequences

A geometric series is one in which there is a common ratio between each term and the next. An example would be 2+ 4+ 8+16… Each term is twice the term before it. Another example is 3+ 6+ 12+ 24… Again, each term is twice the term before it. Just like with arithmetic series and sequences, it doesn't matter what the first term is, we just need to find out if there is a common ratio in order to determine if it is a geometric series. We generally call the first term b_1. The notation for specific terms and sums of geometric sequences is exactly like that of arithmetic sequences. We also have a formula for finding a particular term of a geometric series, and it is

$$b_n = b_1*r^{(n-1)}$$

If we want to know the sum of the first n of these terms, we can use the formula

$$S_n = b_1*(\frac{1-r^n}{1-r})$$

If the common ratio is between -1 and +1, as in the sequence $\{1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8} \ldots\}$, we can find the *infinite sum*, and this is given by

$$S_\infty = \frac{b_1}{1-r}$$

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Examples

Let's look at a couple examples. If we are given a series like: -1 + 3 + 7 + 11 + …, we first need to find out what kind of series it is. Let's pick some terms (we need at least three) and see if it looks like a geometric or arithmetic series. If we try to find the common ratio, we need to pick two adjacent terms, like 3 and 7. The ratio here is $\frac{7}{3}$ and if we pick two other terms, like 7 and 11, the ratio is $\frac{11}{7}$, which is not the same ratio. So let's try to see if it is arithmetic: 3-(-1) = 4, and 7-3 = 4, so it turns out that this one is an arithmetic series. The common difference is 4, and the first term is -1. We could find the 10^{th} term using the formula,

$$a_{10} = -1 + (10-1)*4 = -1 + 36 = 37$$

If we have a sequence like -1, 2, -4, 8, -16…, we do the same thing as before. We must first figure out what kind of series it is. If we try to find a common ratio, we get $\frac{8}{-4}$ =-2, and $\frac{-4}{2}$ = -2, so we do have a common ratio of -2, which means that this is a geometric sequence. It doesn't matter which pairs of number we pick, so long as the numbers in the pairs are next to each other. If we wanted to find the 7^{th} term in this sequence, we use the formula

$$b_7 = -1(-2)^{(7-1)} = -1(64) = -64.$$

A series like 8 + 4 + 2 + 1 + … is another good example. The numbers are getting smaller in this case, but our analysis stays the same. Step one is to find out what type of series this is. 8-4 = 4, but 4-2 = 2, so this can not be an arithmetic series. However, $\frac{4}{8} = \frac{1}{2}$ and $\frac{2}{4} = \frac{1}{2}$, so it is a geometric series, and the common ratio is $\frac{1}{2}$. When determining the common ratio, it is important the numerator be the term after whatever term we put in the denominator. Had we done it the other way, we would have gotten a common ratio of $\frac{8}{4}$ = 2, which is not the number that we multiply by to get the next term in the series. If we wanted to find the 11^{th} term in the series, we simply use the formula again.

$$b_{11} = 8*(\frac{1}{2})^{(11-1)} = 8*(\frac{1}{1024}) = \frac{1}{128}$$

Because our common ratio is between -1 and +1, we can find the infinite sum for this example. $S_\infty = \frac{8}{1-\frac{1}{2}} = \frac{8}{\frac{1}{2}} = 16$. What this means is that if we keep adding all the terms of this sequence, even though there are infinitely many of them, our end result is 16.