# Partial Fraction Decomposition

Many times, it is possible to split a complicated rational expression into a sum of smaller, simpler expressions. This sum is called the *partial fraction decomposition* of the original rational expression. Setting up the decomposition depends on the factors of the denominator, but the process of solving the decomposition is the same.

In general, the decomposition will have one fraction for each factor in the denominator. If the factor is *linear*, the numerator of the fraction is just an unknown constant, like A. If the factor is *quadractic*, the numerator will have a linear form, like A$x$+B.

To solve a decomposition, we solve for the unknown numerators. Let's look at a simple example.

Ex. Find the partial fraction decomposition of $\dfrac{x+14}{(x-4)(x+2)}$.

Step 1: Set up the partial fraction decomposition with the unknown constants A, B, C, etc, in the numerator of the decomposition.

$$\therefore \frac{x+14}{(x-4)(x+2)} = \frac{A}{x-4} + \frac{B}{x+2}$$, and our goal is to find A and B.

Step 2: Multiply both sides of the resulting equation by the least common denominator.

$$\therefore (x-4)(x+2)\frac{x+14}{(x-4)(x+2)} = (x-4)(x+2)\left(\frac{A}{x-4} + \frac{B}{x+2}\right)$$

Step 3: Simplify both sides of the equation. Every denominator should cancel out.

$$\therefore (\cancel{x-4})(\cancel{x+2})\frac{x+14}{(\cancel{x-4})(\cancel{x+2})} = (\cancel{x-4})(x+2)\frac{A}{(\cancel{x-4})} + (x-4)(\cancel{x+2})\frac{B}{(\cancel{x+2})}$$

$$\therefore x+14 = (x+2)A + (x-4)B$$

Step 4: Write both sides in descending powers, equate coefficients of the like terms. Use distributive law on our equation from step 3, and then group like term together.

$$x+14 = (A+B)x + (2A-4B),$$

$$\therefore \begin{cases} 1 = \mathrm{A} + \mathrm{B} \\ 14 = 2\mathrm{A} - 4\mathrm{B} \end{cases}$$

Step 5: Solve for the resulting linear system for A, B, C, etc.

Using substitution, we get A = 3 and B = -2. (Check this yourself!)

Step 6: Substitute the value for A and B into step one

$$\therefore \frac{x+14}{(x-4)(x+2)} = \frac{A}{x-4} + \frac{B}{x+2} = \frac{3}{x-4} - \frac{2}{x+2}$$

We now have our partial fraction decomposition! Our original faction has been split into two smaller fractions.

There is an alternate method we could use in place of steps 4 and 5. We can substitute values of $x$ that make unknown coefficients disappear. In our example,

Steps 4 and 5(alternate): Plug in -2 for $x$, in order to make the term $(x+2)A$ disappear. This gives us

$$\therefore (-2)+14 = (-2+2)A + (-2-4)B$$

$$\therefore 12 = -6B$$

$$\therefore B = -2$$

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Similarly, we can plug in 4 for $x$ to make the term $(x-4)B$ disappear. Solving the resulting equation gives us A=3, just like in our original method.
Many people find this alternate method easier, but unfortunately, it does not always work.

Let's look at a larger, more complicated example. This example has a *repeated* linear factor and a quadratic factor:

Ex: $\dfrac{x^3-x^2+4x-1}{x^2(x^2+1)}$ The denominator has a linear factor, $x$, repeated twice, and a quadratic factor, $x^2+1$.

Step 1: Set up the partial fraction decomposition with the unknown constants A, B, C, etc, in the numerator of the decomposition.

$$\frac{x^3-x^2+4x-1}{x^2(x^2+1)}=\frac{A}{x}+\frac{B}{x^2}+\frac{Cx+D}{x^2+1}$$

Since the linear factor is repeated twice, it needs two fractions, one with an exponent of 1, another with an exponent of 2.

Step 2: Multiply both sides of the resulting equation by the least common denominator.

$$(x^2)(x^2+1)\frac{x^3-x^2+4x-1}{x^2(x^2+1)}=(x^2)(x^2+1)\frac{A}{x}+(x^2)(x^2+1)\frac{B}{x^2}+(x^2)(x^2+1)\frac{Cx+D}{x^2+1}$$

Step 3: Simplify both sides of the equation. Every denominator should cancel out.

$$x^3-x^2+4x-1=(x)(x^2+1)A+(x^2+1)B+(x^2)(Cx+D)$$

Step 4: Write both sides in descending powers, equate coefficients of the like terms. We will have to use distributive law to simplify our equation form step 3, and then group like terms together.

$$x^3-x^2+4x-1=x^3(A+C)+x^2(B+D)+x(A)+B \text{ (Check this for yourself!)}$$

$$\therefore \begin{cases} 1=A+C \\ -1=B+D \\ 4=A \\ -1=B \end{cases}$$

Step 5: Solve for the resulting linear system for A, B, C, etc.

Using substitution, we find that A=4, B=-1, C=-3, and D=0. (Check this for yourself!)

Step 6: Substitute the value for the constants into step one.

$$\frac{x^3-x^2+4x-1}{x^2(x^2+1)}=\frac{A}{x}+\frac{B}{x^2}+\frac{Cx+D}{x^2+1}=\frac{4}{x}+\frac{-1}{x^2}+\frac{-3x+0}{x^2+1}$$

We now have our partial fraction decomposition! In many situations, it will be easier to work with these simpler fractions than our original, complicated fraction.

Unfortunately, this is an example where our alternate method will not get us very far. For this reason, it is important to know more than one method of solving partial fraction decompositions.