Complex Numbers

Introduction to Complex Numbers

The idea of complex (or imaginary) numbers is very useful in many areas. Our discussion of these numbers begins with defining $i = \sqrt{-1}$ so that $i^2 = -1$. In general (but not always), complex numbers have a real part and an imaginary part. They are written in several forms, the most common being

$$z = a + bi$$

where z is the whole complex number, a is the real part, and bi is the imaginary part.

Operations on Complex Numbers

Addition & Subtraction: Two complex numbers, a + bi and c + di, are added and subtracted by simply combining the real parts and imaginary parts separately, as follows:

$$(a + bi) + (c + di) = (a + c) + (b + d)i$$
$$(a + bi) - (c + di) = (a - c) + (b - d)i$$

Multiplication: Two complex numbers are multiplied using a process just like F.O.I.L.

$$(a + bi) * (c + di) = ac + adi + bci + bdi^2 = (ac - bd) + (ad + bc)i$$

Division: To divide complex numbers, we need to first learn *complex conjugates.* Two complex numbers are called complex conjugates if they have the same real part and the same imaginary part, with only one of the signs changed. (It is far more common to change the sign of the imaginary part, but either will work). Thus, 1 + i and 1 - i are complex conjugates, and so are 1 + i and -1 + i. Note that although 1 – i and -1 + i are both complex conjugates of the same number, they are definitely not equivalent to each other. The advantage of complex conjugates is that if a complex number is multiplied by its complex conjugate, the product is *always a real number.*

To divide two complex numbers, we first write the division as a fraction, and then multiply the numerator and denominator by the complex conjugate of the *denominator.* We foil out the top and bottom, and separate the real part from the imaginary, and express our answer in the form a + bi.

$$\text{Ex. } (1 + i) \div (2 - i) = \frac{1+i}{2-i} = \frac{1+i}{2-i} * \frac{2+i}{2+i} = \frac{1-3i}{5} = \frac{1}{3} - \frac{3}{5}i \text{ or } \frac{1}{3} - \frac{3i}{5}$$

Multiplication and division, as well as some other operations are much more easily performed using other forms of complex numbers, as we will see later on.

The Trigonometric Form of Complex Numbers

Any complex number, z = a + bi, can be written in its equivalent trigonometric form, $z = r*(\cos\theta + i*\sin\theta)$. In this form, r is the magnitude, and θ is the argument.

The conversions are:

$$\theta = \tan^{-1}(b/a) \qquad\qquad x = r * \cos\theta$$

$$r = \sqrt{a^2 + b^2} \qquad\qquad y = r * \sin\theta$$

*Note: Be careful when finding θ, because $\tan^{-1}$ is only defined for $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$, but complex numbers can be anywhere from $0 < \theta < 2\pi$. Because of this, we should think of the answer from the calculator as the reference angle, and use what we know about the signs of the real and imaginary parts.

As an example, z = -1 + i is in the second quadrant, but $\tan^{-1}(\frac{1}{-1}) = -45°$, which is in the fourth quadrant and cannot be our angle. If we use the reference angle, 45°, and the fact that the angle we want is in the second quadrant, we get 180°-45° = 135°.

<u>The Exponential Form</u>

There is a third form used to represent complex numbers called the exponential form, and it is related to the trigonometric form by Euler's Identity:

$$e^{i\theta} = \cos\theta + i*\sin\theta, \quad \text{so} \qquad re^{i\theta} = r(\cos\theta + i*\sin\theta)$$

This form is especially useful when we want to raise complex numbers to large exponents. For example, $(1 + i)^{100}$ would be extremely cumbersome, even with the binomial theorem. If we instead convert (1 + i) to its exponential form, $r = \sqrt{1^2 + 1^2} = \sqrt{2}$ and $\theta = \tan^{-1}(1) = \pi/4$

$(1 + i)^{100} = (\sqrt{2}\, e^{i\pi/4})^{100} = 2^{50} * e^{i25\pi} = 2^{50} * e^{i\pi} = 2^{50}*(-1) = -2^{50}$, which is much easier to work with.

It is very difficult to try to add or subtract complex numbers in this form, but multiplication and division are very easy. In multiplication, multiply the magnitudes and add the arguments. In division, divide the magnitudes and subtract the arguments.

There is another useful formula involving the exponential form known as *DeMoivre's Theorem*. It says

$$(\cos\theta + i*\sin\theta)^n = \cos(n\theta) + i\sin(n\theta)$$

This relation is due to the fact that $(\cos\theta + i*\sin\theta)^n = (e^{i\theta})^n = e^{in\theta}$
This can be used to find equivalent expressions for functions such as $\cos(3\theta)$.
$\cos\theta$ is the *real* part of $\cos\theta + i*\sin\theta$, so $\cos 3\theta$ is the *real* part of $(\cos\theta + i*\sin\theta)^3$
The right side can be expanded using FOIL, and we will get $\cos 3\theta = \cos^3\theta - 3\cos\theta\sin\theta$.