

Probability

Probability is a huge topic and we won't be able to cover everything here, but we can get to some basics. First of all, we may classify most probabilities into one of two categories: classically (theoretically) determined probabilities and experimentally determined probabilities.

Classically determined probabilities require that each possible event have an equal chance of occurring.

$$P(A) = \frac{\text{number of ways event A may occur}}{\text{total number of different outcomes possible}}$$

The notation "P(A)" means "the probability of event A."

For example, let's say I want to know the probability of rolling a "4" with a fair die.



$$P(\text{rolling a "4"}) = \frac{\text{number of ways that a "4" may occur}}{\text{number of different outcomes}} = \frac{1}{6}$$

Note that a "4" can only occur in one way, since there is only one "4" on a die. Note also that the number of different outcomes is 6, since a die has six sides.

Experimentally determined probabilities are based on many observations or trials. We would perform (or observe) several trials and count how many times we get a desired result.

$$P(A) = \frac{\text{number of times event A occurs}}{\text{total number of trials}}$$

For example, I may find the probability of making a free throw in basketball by going out and attempting a free throw many times. Then the probability is:

$$P(\text{making a free throw}) = \frac{\text{number of times I make a free throw}}{\text{total number of attempts}}$$



This is really an approximation because if I go out the next day and do the same experiment, I may get a different probability. It makes sense that the more attempts I try, the closer the experimental probability will get to the actual probability. To understand this idea that more trials leads to a better approximation, imagine the experiment of flipping a coin.

Because there are two sides (heads/tails) on a fair coin, the theoretical probability for getting "tails" is $\frac{1}{2}$ or 0.5. If I only flip the coin once and happen to get tails, I would conclude that $P(\text{tails}) = 1$. This is obviously not correct. I must flip the coin many more times to get a better approximation of the theoretical probability. Maybe if I flip the coin 100 times, I get 55 tails. I would conclude that $P(\text{tails}) = 0.55$, which is a lot closer to 0.5. If I flip the coin 10,000 times, I would expect to get an experimental value very close to the theoretical value of 0.5.

For some probabilities, it is rather easy to find a theoretical value (like the probability of getting “tails” when flipping a coin). Other probabilities lend themselves to the experimental approach and it is difficult and very complicated to find a theoretical value. Sometimes, one can find a theoretical value and then try to verify it experimentally (as in our experiment of flipping a coin several times).

What is the probability of rolling a “7” with a single fair die? The answer is zero, because there isn’t a “7” on the die. So, the probability of an impossible event occurring is zero. What is the probability of rolling a number between one and six, inclusive? The answer here is one, since all the numbers on a die are between one and six, inclusive. So, the probability of an event that is certain to occur is one. All probabilities therefore, must lie between zero and one, inclusive. We can say this symbolically as $0 \leq P(A) \leq 1$.



For future examples, we will refer to a standard deck of cards. Allow me to define the contents of the deck for those who have never played cards before. Even if you know the contents, it will be nice to have a table of information to refer to.

<u>Color</u>	<u>Suit</u>			
red	hearts	(2, 3, 4, 5, 6, 7, 8, 9, 10, Jack, Queen, King, Ace)	= 13 red hearts	
red	diamonds	(2, 3, 4, 5, 6, 7, 8, 9, 10, Jack, Queen, King, Ace)	= 13 red diamonds	
black	spades	(2, 3, 4, 5, 6, 7, 8, 9, 10, Jack, Queen, King, Ace)	= 13 black spades	
black	clubs	(2, 3, 4, 5, 6, 7, 8, 9, 10, Jack, Queen, King, Ace)	= <u>13 black clubs</u>	
52 cards total				

Addition Rule

To find probabilities of the form $P(A \text{ or } B)$, we may employ the addition rule. The probability of A or B will include the probability of event A occurring, event B occurring or both events A and B occurring at the same time on one trial. The addition rule states:

$$P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B) \longrightarrow$$

We must subtract the probability of A and B occurring at the same time because we don’t want to count these outcomes again. They were already included in $P(A)$ (counted once) and $P(B)$ (counted twice), so we must subtract $P(A \text{ and } B)$ to avoid this double-counting.

Let’s use an example from our standard deck of cards.

What is the probability of being dealt a king or a diamond from a shuffled deck? If we apply the addition rule to our example (since we are looking for a king or a diamond, the addition rule applies), we get the following:

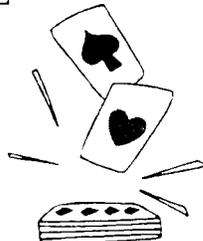
$$P(\text{king or diamond}) = P(\text{king}) + P(\text{diamond}) - P(\text{king and diamond})$$

$$= \frac{4}{52} + \frac{13}{52} - \frac{1}{52}$$

There are 4 kings in the deck and 52 total cards.

There are 13 diamonds and 52 total cards.

There is one king of diamonds and 52 total cards.

$$= \frac{4 + 13 - 1}{52} = \frac{16}{52} = \frac{4}{13} \approx 0.308$$


Another example: What is the probability of being dealt a heart or a spade?

$$P(H \text{ or } S) = P(H) + P(S) - P(H \& S)$$

$$= \frac{13}{52} + \frac{13}{52} - \frac{0}{52} = \frac{26}{52} = \frac{1}{2} = 0.5$$

Note that there aren't any cards that are both hearts and spades at the same time. Two events that cannot occur at the same time are said to be mutually exclusive.

Multiplication Rule

The multiplication rule applies to repeated experiments or trials. This means that we are no longer being dealt only one card, but rather several cards. Before we look at some examples, we need to discuss independent vs. dependent events. Two events are said to be independent if the occurrence of one event does not have an effect on the likelihood of the other event occurring. Two dependent events are events where the occurrence of one does influence the probability of the other occurring.

As an example, consider the experiment of being dealt a couple cards from our deck. If I put the first card I'm dealt back into the deck before being dealt the second card (I'm replacing the first card back into the deck), then the two trials are independent (I am randomly selecting from 52 cards in each trial). However, if I do not replace the first card before being dealt the second, then the two trials are dependent (in the second trial, I only have 51 cards left to pick from, since the one selected on the first trial is not replaced). Remember, in the context of the multiplication rule, with replacement means independent and without replacement means dependent.

So for independent events, the multiplication rule is

$$P(A \text{ and } B) = P(A) * P(B)$$

Note here that P(A and B) refers to the probability of A on one trial and B on the next trial, not the probability of A and B occurring simultaneously on one trial, which is what this notation referred to with respect to the addition rule.

And for dependent events, the multiplication rule is

$$P(A \text{ and } B) = P(A) * P(B|A)$$

P(B|A) means the probability of event B occurring, given that event A has already occurred.

Let's try some examples using the multiplication rule.

What is the probability of being dealt a "6" and a "7" (in that order) with replacement?
Remember that "with replacement" means independent events.

$$P(6 \text{ and } 7) = P(6) * P(7) = \frac{4}{52} * \frac{4}{52} = \frac{16}{2704} = \frac{1}{169} \approx 0.00592$$



What is the probability of being dealt two hearts without replacement?

$$P(H \text{ and } H) = P(H) * P(H|H) = \frac{13}{52} * \frac{12}{51} = \frac{156}{2652} = \frac{1}{17} \approx 0.0588$$

The notation means the probability of being dealt a heart on the second trial, given that you've already been dealt a heart on the first trial.

If one heart has been removed from the deck for the second trial, then there are only 12 hearts left and 51 cards left.

What is the probability of being dealt a "6" and a "7"? Note that this problem does not say in which order you are dealt these two cards, nor does it say with or without replacement. We must use a little common sense to determine these things. When playing cards, one rarely puts a card they are dealt back into the deck before being dealt the next card, so we will interpret this problem as a "without replacement" problem. Also, since an order is not given for the two cards, we will have to consider the possibility of being dealt a "6" and then a "7" or a "7" and then a "6." Both of these situations are implied in the question, even though they are not stated explicitly. Often problems will be worded in this fashion, and we need to realize which outcomes need to be included in our calculations. Keep this in mind when interpreting a problem.

To solve this problem, let's consider first the probability of being dealt a 6 and then a 7:

$$P(6 \text{ and } 7) = P(6) * P(7|6) = \frac{4}{52} * \frac{4}{51} = \frac{16}{2652} = \frac{4}{663}$$

There are still four 7's in the deck, but only 51 cards left total.

And now the probability of a 7 and then a 6:

$$P(7 \text{ and } 6) = P(7) * P(6|7) = \frac{4}{52} * \frac{4}{51} = \frac{16}{2652} = \frac{4}{663}$$

Now we must add these two probabilities together. Remember the addition rule. A {"6 and then a 7"} or a {"7 and then a 6"} will satisfy the terms of our experiment, and the "or" tells us to use the addition rule. It is impossible to get both situations at the same time, so these two must be mutually exclusive.

$$P(\text{"6 and 7" or "7 and 6"}) = P(6 \text{ and } 7) + P(7 \text{ and } 6) = \frac{4}{663} + \frac{4}{663} = \frac{8}{663} \approx 0.0121$$

Let's try some more examples. What is the probability of being dealt five cards and all of them being the same suit?

This problem will employ the multiplication rule since we are talking about repeated trials. Keep in mind that the problem does not specify which suit the cards have to be, the only requirement is that they are all the same suit. What this suggests is that the first card may be any suit. Then the subsequent four cards must match the suit that the first card "decided." The probability of the first card being any suit is one, since all the cards in the deck have a suit.

$$P(5 \text{ cards with same suit}) = 1 * \frac{12}{51} * \frac{11}{50} * \frac{10}{49} * \frac{9}{48} \approx 0.00198$$



The first card can be any suit, so the probability for this trial is one.

After the first card "decides" what all the rest have to match, there are 12 cards left of that suit and 51 total cards. The rest of the trials are also dependent on the outcomes in previous trials.

We could have considered a similar case, where the suit was specified. For example, we could have asked for the probability of being dealt five clubs from our deck. The only thing that would change in this problem is that the first probability would be 13/52, since this first draw no longer gets to "decide" what all the others must match. Now the first draw is under the constraint that it must be a club. This probability would work out like this:

$$P(5 \text{ clubs}) = \frac{13}{52} * \frac{12}{51} * \frac{11}{50} * \frac{10}{49} * \frac{9}{48} \approx 0.000495$$

It makes sense that this is a smaller probability than that of getting 5 cards of the same suit. Since we are looking for a more selective outcome, we expect a lower probability.

This next problem is more complicated, and your textbook may not contain examples like it. If you're feeling comfortable so far, go ahead and take a look and see if you can follow along ...

Let's say that we're dealt three cards, and I want to know the probability of getting at least two cards of the same suit.

In this situation, we have several things to consider. First, the problem does not state a specific suit, so the first card can be any suit. We also have to consider the phrase "at least two" which suggests that either two of the cards have matching suits, or all three match. There is also the issue of the order we are dealt the cards if only two match. We could have our first two cards matching, or the last two matching, or the first and the last card matching. Let's break this problem down into smaller pieces.

$$P(\text{at least two out of three cards are of the same suit}) = P(\text{two cards have matching suits} \text{ or } \text{all three match}) \\ = P(\text{two cards match suit}) + P(\text{all three match})$$



These two events are mutually exclusive, but they are not simple events.

P(two cards match suit) → To figure this out, we must consider the order of the cards. Remember that the first card can be any card, but our matching suit pair could be the first and second card, the second and third card, or the first and third card. So, there are three situations that would satisfy the requirements of this experiment.

1	*	$\frac{12}{51}$	*	$\frac{39}{50}$	=	$\frac{78}{425}$
any suit		match 1 st		no match		
1	*	$\frac{39}{51}$	*	$\frac{12}{50}$	=	$\frac{78}{425}$
any suit		no match 1 st match 3 rd		match 2 nd		
1	*	$\frac{39}{51}$	*	$\frac{12}{50}$	=	$\frac{78}{425}$
any suit		no match		match 1 st		

If we use addition rule and add these up, we will get the probability of exactly two cards matching suit.

P(two cards match suit) = $\frac{234}{425}$

The second card must match the first. There are 12 cards that match the first card's suit. The third card must not match. There are three non-matching suits of 13 cards each, so 39 cards that don't match. This same reasoning applies to the other possible outcomes listed.

P(all three cards match suit) → This probability isn't as difficult to compute. Again, remember that the first card may be any suit, but the next two have to match whatever that first card "decides."

1	*	$\frac{12}{51}$	*	$\frac{11}{50}$	=	$\frac{22}{425}$
any suit		match		match		

P(all three match suit) = $\frac{22}{425}$

So, the probability of getting at least two cards of the same suit when three are dealt is...

P(at least two out of three cards are of the same suit) = P(two cards have matching suits or all three match)



$$\begin{aligned}
 &= P(\text{two cards match suit}) + P(\text{all three match}) \\
 &= \frac{234}{425} + \frac{22}{425} = \frac{256}{425} \approx \text{0.602}
 \end{aligned}$$