Logarithms and Exponentials

A logarithmic function is the inverse of an exponential function, and an exponential function is the inverse of a logarithmic function. For example,

\[ f(x) = 2^x \] \hspace{1cm} \text{inverse} \hspace{1cm} \frac{1}{f(x)} = \log_2 x \]

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Note: An exponential function is one with a variable in the exponent. So, \( f(x) = b^x \) (where \( b \) is a positive constant, \( b \neq 1 \)) is the exponential function, base \( b \). The function \( f(x) = x^2 \) is NOT an exponential function.

How to read logarithms (“logs”):

When you have \( \log_b x \) ... it is read, "the log, base \( b \), of \( x \)."

Or when you have \( \log_2 x^4 \) ... it is read, "the log, base 2, of \( x \) to the power of 4."

Note: In the expression \( \log_b c \), \( b \) is the “base” and \( c \) is called the “argument.” (\( b > 0, c > 0 \))

Converting LOGARITHMS into EXPONENTIALS and EXPONENTIALS into LOGARITHMS:

“\( a \) is equal to the log, base \( b \), of \( c \)” \hspace{1cm} “\( b \) to the power of \( a \) is equal to \( c \)”

\[ a = \log_b c \] \hspace{1cm} \text{converts to} \hspace{1cm} \frac{1}{a} = c \]

Note: In the logarithmic equation, the base is \( b \). This corresponds to the base \( b \) in the exponential equation. In the exponential equation, the exponent is \( a \), and this corresponds to what the entire log is equal to in the logarithmic equation. The logarithm is an exponent. If we can remember how these “positions” of \( a, b, c \) convert from logarithms to exponentials and back again, then rewriting logs as exponentials or exponentials as logs won’t be a problem.

\[ \text{ex: } \log_2 9 = 3 \] \hspace{1cm} \text{converts to} \hspace{1cm} x = \log_3 9 \]

\[ y = \log_3 27 \] \hspace{1cm} \text{converts to} \hspace{1cm} 3^y = 27

LOGARITHMIC and EXPONENTIAL Equality:

If ... \[ b^x = b^y \] \hspace{1cm} The bases "\( b \)" are the same (\( b \neq -1, 0, 1 \)).
then ... \[ x = y \] \hspace{1cm} We may equate exponents.

If ... \[ \log_a x = \log_a y \] \hspace{1cm} The bases "\( a \)" are the same (\( x, y, a > 0 \)).
then ... \[ x = y \] \hspace{1cm} We may equate the arguments.

\[ \text{ex: } 4^{x+1} = 4^{2x} \] Since the bases are both “4,” we may equate exponents.
\[ x + 2 = 4 \]
Answer. The argument must be greater than zero, so \( x + 2 > 0 \). ... 2 + 2 > 0 ...
... 4 > 0 ...

\[ 1 = x \] Answer.
Logarithmic Properties: The following is a list of some logarithmic properties that may be useful.

| PRODUCT RULE for LOGS | \( \log_b(MN) = (\log_b M) + (\log_b N) \) | For any positive numbers \( M, N, \) and \( b \) (\( b \neq 1 \)).
ex: \( \log_2 (xy) = (\log_2 x) + (\log_2 y) \)

| QUOTIENT RULE for LOGS | \( \log_b \frac{M}{N} = (\log_b M) - (\log_b N) \) | For any positive numbers \( M, N, \) and \( b \) (\( b \neq 1 \)).
ex: \( \log_{10} \frac{x^2}{y} = (\log_{10} x^2) - (\log_{10} y) \)

| POWER RULE for LOGS | \( \log_b M^p = p (\log_b M) \) | For any positive number \( M \) and \( b \) (\( b \neq 1 \)), and any real number \( p \).
ex: \( \log_2 25^x = x (\log_2 25) \)

| LOG of a BASE raised to a POWER | \( \log_b b^k = k \) | For any base \( b \).
ex: \( \log_2 2^x = x \)

| CHANGE of BASE | \( \log_b M = \frac{\log_a M}{\log_a b} \) | For any logarithmic bases \( a \) and \( b \), and any positive number \( M \).
ex: \( \log_3 5 = \frac{\ln 5}{\ln 3} \)

Note also: You CANNOT take the log of a negative number.
\( \log_{10} (-4) = \text{undefined} \)

One more thing: Remember “log” is not distributive ...

\( \log (M \pm N) \neq \log M \pm \log N \)
\( \log (MN) \neq (\log M)(\log N) \)
\( \log \left( \frac{M}{N} \right) \neq \frac{\log M}{\log N} \)

Note: When using the change of base formula, we will often change to base 10 (so in the change of base formula, \( a = 10 \)). In this way, you can use the \( \log \) button on your calculator to solve the problem. A log with base 10 is called the “common log” and is written without the ten, so \( \log_{10} 5 \) is written \( \log 5 \). The “natural log” is another log that comes up often. The natural log is just a log with base “\( e \)” (“\( e \)” is a constant equal to approximately 2.718281828). Natural logs use \( \ln \) instead of \( \log_e \), so \( \log_{e} 4 \) is written \( \ln 4 \). The \( \ln \) button is also on most calculators, so you could change to base “\( e \)” if you choose.

ex: \( \log x = 3.8 \) Rewrite the common log as a power of ten.
\( 10^{3.8} = x \) Exact answer.
\( 6309.57 \approx x \) Approximate answer. Checks since \( 6309.57 > 0 \)

ex: \( \ln x = 2 \) Rewrite the natural log as a power of e.
\( e^2 = x \) Exact answer.
\( 7.39 \approx x \) Approximate answer. Checks since \( 7.39 > 0 \)